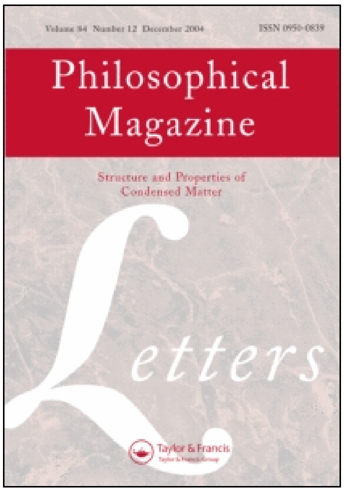

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Velocity dependence of shear localisation in a 2D foam

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We give an exact formula for the velocity profile of shear localisation in a 2D foam, represented by a continuum model that incorporates a Herschel–Bulkley constitutive relation and wall drag. A more approximate treatment provides a relation between the localisation length and the boundary velocity as a power law whose exponent is explicitly determined by the input parameters of the model. This is corroborated, and its conditions for validity are clarified, by the analysis of various expansions of the exact solution. The general consequences are consistent with the recent findings of Katgert and co-workers (G. Katgert, M. Möbius, and M. van Hecke, available from http://arxiv.org/abs/0711.4024 [cond-mat.soft].

1. Introduction

Two-dimensional (2D) foams have been found to exhibit strong shear localisation at a moving boundary, in most cases [1–4]. The analysis presented here follows from a previous one undertaken by us, based on a continuum description [5–7]. It is prompted by a recent important contribution by Katgert et al. [8].

Our previous analysis combined a Bingham constitutive law and a linear drag force at the containing wall(s) of the 2D foam. Its essential results were an exponential localisation and a decay length that is independent of boundary velocity. It was acknowledged that the linear forms in the constitutive law and drag term were crude approximations, made in the interests of simplicity and in the light of uncertainty regarding more realistic forms. Here, as in [8], we introduce power laws, but we allow these to be entirely general, so that the two power law indices characterise any particular model of this type. We succeed in deriving exact velocity (or shear rate) profiles, but these analytic forms are clumsy. Accordingly, we derive approximate formulae for the variation of the localisation length (or shear band width) with boundary velocity. Since in general the profile is not exponential, the precise definition of this width is bound to be somewhat arbitrary.

It will be interesting to comment on the quasistatic limit of these results, since a number of simulations have been made in that spirit. However, at this stage no claim is made to any comprehensive reconciliation of the various experiments [1–4], cellular

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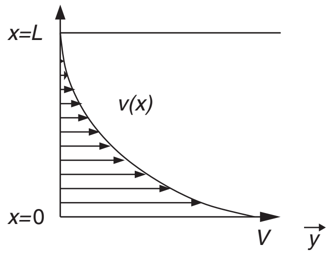

Figure 1. In a 2D shear cell, the boundary at $x = 0$ moves with velocity V while that at $x = L$ is fixed. In the theoretical model considered here, the shape of the steady-state velocity profile $v(x)$ is determined by the exponents in both the Herschel–Bulkley model and the drag force law.

simulations [9,10] and continuum model calculations [5–7] that are already published. Rather, we offer a simple framework towards that end.

2. Steady-state profile of velocity

We adopt the geometry of Figure 1, in order to be consistent with [5], although this causes some minor difficulties, as we shall see. A sample of 2D foam when subjected to shearing motion is accommodated between two parallel straight boundaries at $x = 0$, L. The frame of reference is that in which the containing plates are fixed (a seemingly obvious point but important in this context). The boundary at $x = 0$ moves with velocity V while the other is fixed. In some of the experiments cited above, a circular geometry was used [1–3].

The problem is much simplified by concentrating entirely on the steady-state velocity profile $v(x)$, and we shall not attempt to describe transients [5].

We will adopt the following definition for the localisation length l, on the grounds that it is consistent with the natural choice for exponential localisation, and convenient for experiment,

$$v(l) = V/e. \tag{1}$$

As said before [5–7], we proceed by imposing the condition of force balance between stress gradient and drag force on an element at x. We use the Herschel–Bulkley equation as the constitutive law. This is usually written as

$$\sigma = \sigma_y + c_v \dot{\gamma}^a, \tag{2}$$

where σ and σ_y denote stress and yield stress, respectively, c_v is the viscosity of the Herschel–Bulkley fluid (also called *consistency*) and a is the Herschel–Bulkley exponent. The strain rate $\dot{\gamma}$ is given by $\dot{\gamma} = \mathrm{d}v/\mathrm{d}x$. (Note that in two dimensions stress has the dimension of a force divided by a length.)

In the present case, σ, σ_y and $\dot{\gamma}$ are all *negative*, so Equation (2) should be rewritten as

$$\sigma = \sigma_y - c_v|\dot{\gamma}|^a, \tag{3}$$

where c_v is positive.

The wall drag force per unit area is

$$F = -c_d v^b, \tag{4}$$

with a positive drag force coefficient c_d and an exponent b. This in principle raises a similar problem if v is negative, but it is positive everywhere in the case considered.
The generalisation of Equation (3) of reference [5] is then

$$-c_v \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{\mathrm{d}v}{\mathrm{d}x}\right)^a = c_d v^b. \tag{5}$$

In accordance with Figure 1, the boundary conditions are:

$$v = V \text{ at } x = 0 \quad \text{and} \quad v = 0 \text{ at } x = L \tag{6}$$

and we require $\mathrm{d}v/\mathrm{d}x$ to be continuous everywhere.

In order to facilitate the search for an analytic solution, we will express Equation (5) in terms of dimensionless variables $\tilde{x} = x/L$ and $\tilde{v} = v/V$. This results in

$$-\xi \frac{\mathrm{d}}{\mathrm{d}\tilde{x}}\left(-\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{x}}\right)^a = \tilde{v}^b, \tag{7}$$

where the dimensionless parameter ξ is given by

$$\xi = \frac{c_v}{c_d}\frac{V^{a-b}}{L^{a+1}}. \tag{8}$$

We obtained an implicit analytic solution for $\tilde{v}(\tilde{x})$ with two constants A_1 and A_2 (to be determined by the boundary conditions of Equation (6)),

$$\left(\frac{1+a}{\xi a(1+b)}\right)^{1/(1+a)}(A_2 - \tilde{x}) = A_1^{-1/(1+a)}\tilde{v}(\tilde{x})\, {}_2F_1\left(\frac{1}{1+b}, \frac{1}{1+a}, 1 + \frac{1}{1+b}, -\frac{\tilde{v}(\tilde{x})^{1+b}}{A_1}\right), \tag{9}$$

where ${}_2F_1$ is Gauss's hypergeometric function (Equation (15.3.1) of [11]). We have also verified this solution using mathematical software Mathematica and Maple.[1]

Requiring $\tilde{v}(1) = 0$ fixes the constant A_2, as $A_2 = 1$. The constant A_1 can then be found by a root-finding method to match the other boundary condition, $\tilde{v}(0) = 1$. Figure 2 shows the localisation of the velocity profile close to the moving boundary for the case $a < b$.

Note that the boundary condition $\tilde{v}(0) = 1$ cannot be matched for arbitrarily low values of ξ in the case $a > b$ where there is a different solution, see Section 5.2.

As shown in section 5, various asymptotic forms can be extracted from Equation (9) for the localisation length. But for the moment, we show how to extract a useful approximation directly from Equation (7) in a simple way.

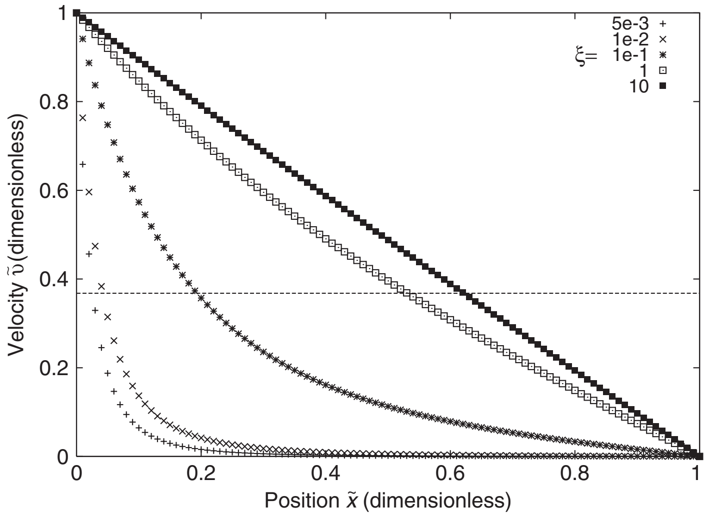

Figure 2. Plot of the analytic solution of Equation (5) that satisfies the boundary conditions of Equation (6). Decreasing the dimensionless parameter ξ leads to increasing velocity localisation. Intersections with the dashed line correspond to $\tilde{v} = 1/e$. The corresponding value of $\tilde{x}$ is taken as localisation length. The data shown is for $a = 0.5$ and $b = 1$.

3. Approximate formula for the localisation length

Here, we use a simple argument from which a formula for the localisation length $l(V)$ conveniently emerges as a power law. We should stress that there is no exact justification for such a form and that the derivation is manifestly 'intuitive'.

Integration of Equation (7) over $\tilde{x}$ from $x = 0$ to $x = l/L$, the range over which $\tilde{v}$ departs significantly from zero, results in

$$-\xi\left(-\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{x}}\right)^{a}\Bigg|_{\tilde{x}=0}^{\tilde{x}=l/L} = \int_0^{l/L} \mathrm{d}\tilde{x}\ \tilde{v}^{b}. \tag{10}$$

Consider the left-hand side. The value of $\mathrm{d}\tilde{v}/\mathrm{d}\tilde{x}$ at $\tilde{x} = l/L$ can be approximated by zero. At $\tilde{x} = 0$, the slope of $\tilde{v}(\tilde{x})$ may be approximated by $-L/l$ in a crude linear approximation of $\tilde{v}(\tilde{x})$ in the range $[0, l/L]$. (Note that the latter value corresponds to $\mathrm{d}v/\mathrm{d}x = -V/l$ in the original units.) The left-hand side of Equation (10) is thus given by $\xi(L/l)^a$.

In the integral on the right-hand side, we approximate the local velocity by its value at the moving boundary, i.e. $\tilde{v} \simeq 1$. Then, this reduces the right-hand side of Equation (10) to l/L.

Equating these approximations for left- and right-hand side results in the following approximate relationship:

$$\frac{l}{L} \simeq \xi^{1/(a+1)}. \tag{11}$$

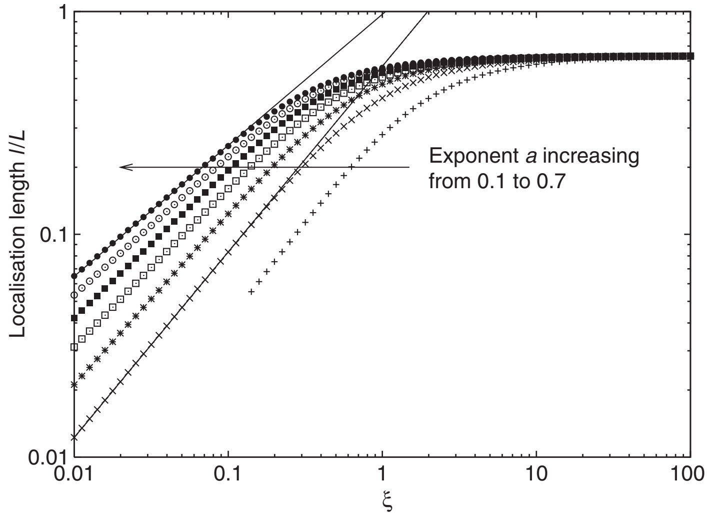

Figure 3. Variation of the non-dimensionalised localisation length l/L as a function of the parameter $\xi = (c_v/c_d)(V^{a-b}/L^{a+1})$ of Equation 8. Here, the drag exponent b was set to $b = 1$ and the Herschel–Bulkley exponent a was varied in equal intervals from 0.1 to 0.7. The solid lines are fits to the power law of Equation (11) with exponents $(1+0.2)^{-1}$ and $(1+0.7)^{-1}$, respectively.

Restoration of the physical units by inserting Equation (8) for ξ finally gives the key result:

$$l \simeq \left(\frac{c_v}{c_d}\right)^{1/(1+a)} V^n \tag{12}$$

where we have defined a localisation exponent n as

$$n = \frac{a-b}{1+a}. \tag{13}$$

To test the validity of Equation (11), we determine the localisation length l (Equation (1)) numerically, using the exact solution for $\tilde{v}(\tilde{x})$ (Equation (9)), see Figure 2.

Its variation as a function of ξ, for a number of values of the Herschel–Bulkley exponent a and a fixed value of the drag exponent $b = 1$, is shown in Figure 3. The localisation length tends to a constant for large values of ξ, but exhibits a power-law behaviour in the limit $\xi \to 0$. The solid lines are fits to Equation (11) in this limit, with the value of the exponent given by $(a+1)^{-1}$.

Figure 4 shows the variation of the localisation exponent n of Equation (13) as a function of the Herschel–Bulkley exponent a. It is in excellent agreement with the exponents obtained from our numerical data.

4. Implications

The formula, Equation (12), that we propose is manifestly approximate, and depends on an arbitrary definition of localisation length. Nevertheless, it should be very useful in

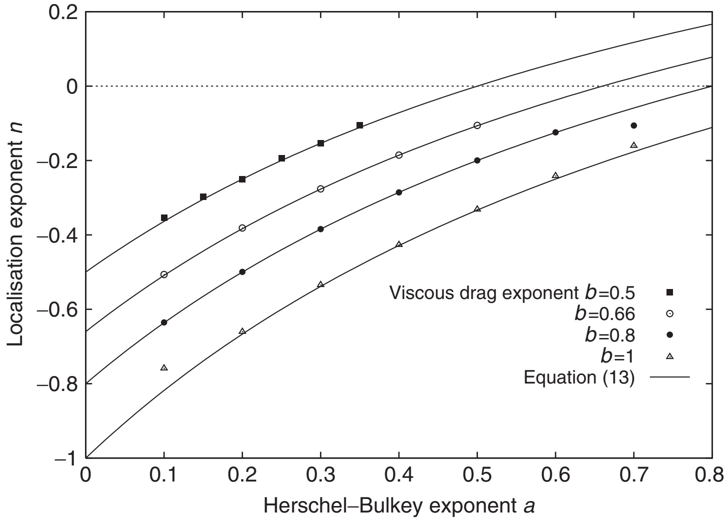

Figure 4. Variation of the localisation exponent n (Equation (13)) with the Herschel–Bulkley exponent a of Equation (3) for four different values of the viscous drag exponent b of Equation (4). The solid lines are our estimates of this variation, as given by Equation (13). Note that $n = 0$ for $a = b$, i.e. there is no velocity dependence of the localisation length l in this case.

further debate on this topic. The special case $a = b$ is interesting, in that it yields a localisation length that is independent of V. The cases in which a is greater than and less than b are qualitatively distinct, in that the localisation length increases (i.e. $n > 0$) or decreases (i.e. $n < 0$) with V, see Figure 4. This may be explained as follows. The localisation length is determined by a balance of the internal dissipation and that due to the external (wall drag) force. When both have linear forms, as in the original model [5], the localisation length was constant. If instead the two terms vary as powers of V, the larger power increasingly dominates as V is increased. Recalling that wall drag tends to localise shear and the internal viscosity de-localises it, we can see how the two behaviours in Figure 5 arise.

5. Exact derivation of asymptotic forms

The localisation length is given by setting $\tilde{v}(l/L) = 1/e$ in our analytical solution, Equation (9),

$$\frac{l}{L} = 1 - \left(\frac{\xi a(1+b)}{1+a}\right)^{1/(1+a)} A_1^{-1/(1+a)}(1/e)\ {}_2F_1\left(\frac{1}{1+b}, \frac{1}{1+a}, 1 + \frac{1}{1+b}, -\frac{(1/e)^{1+b}}{A_1}\right). \quad (14)$$

The constant A_1 is determined from the boundary condition $\tilde{v}(0) = 1$. This amounts to solving

$$\xi = \frac{1+a}{a(1+b)} \frac{A_1}{{}_2F_1\left(\frac{1}{1+\beta}, \frac{1}{1+a}, 1 + \frac{1}{1+b}, -A_1^{-1}\right)^{(1+a)}} \quad (15)$$

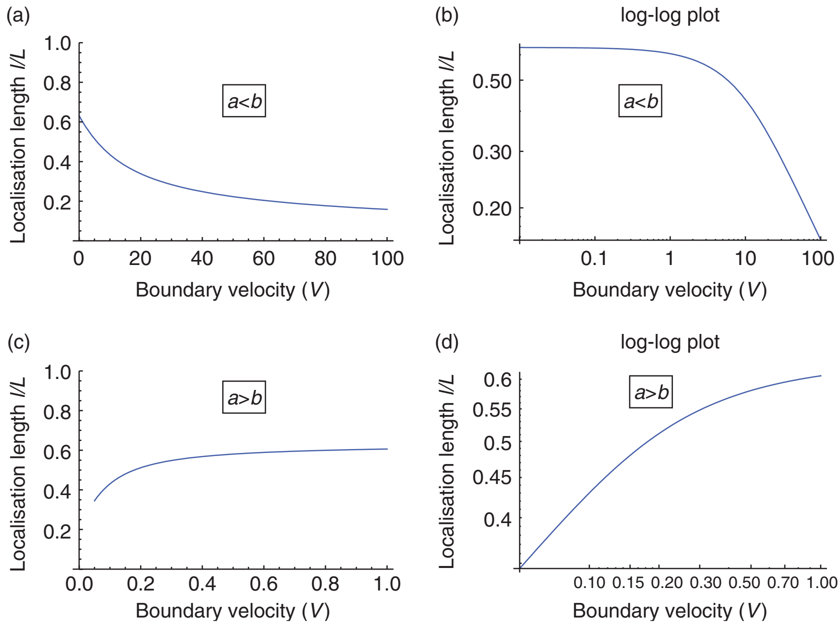

Figure 5. Exemplary calculations for the dependence of localisation length l/L on velocity V for the cases $a < b$ and $a > b$. The sets of parameters were $a=1$, $b=2$ in plots (a) and (b) and $a=2$, $b=1$ in plots (c) and (d). (For this figure we set $\xi = V^{a-b}$, i.e. $(c_v/c_d)L^{-a-1}$ was set to unity).

for a particular choice of ξ, which can generally be done only numerically. (Note that ξ is a monotonically increasing function of A_1 for $\xi > 0$, as is the case here.)

Exemplary plots of localisation length versus velocity, obtained from Equation (14) are shown in Figure 5 for the cases $a < b$ and $a > b$ to illustrate the different qualitative behaviour.

For $a < b$, the localisation length l decreases with the boundary velocity V asymptotically as a power law. In the limit of $V=0$, it tends to a constant. For the definition of l that we use here, this is $L(1-1/e)$. In this limit the profile is simply linear, $v=V(L-x)/L$.

For $a > b$, the localisation length l increases with V, tending to the same constant as $V \to \infty$. For low V there again appears to be power-law behaviour. (However, see the comments in Section 5.2.)

The constant limiting value of l depends on the definition of the localisation length. If we use the alternative definition which sets $1/l$ equal to the logarithmic derivative of v at $x=0$ (as was used in part of the argument of Section 3), this leads to broadly similar behaviour, and the value $l=L$ in the delocalised limit.

5.1. *Expansion for* $\xi \to \infty$

While we cannot invert Equation (15) analytically, it is possible to find the limiting behaviour of the localisation length for $a > b$ and $a < b$ by expanding the equation around $A_1=0$ and ∞.

In the case $A_1 \to \infty$, making use of Equation (15.1.1) of [11] we get

$$\xi = \frac{1+a}{a(1+b)}\left(\frac{1}{2+b} + A_1\right) + O(A_1^{-1}) \tag{16}$$

We write A_1 as a power series in ξ and solve Equation (16) term-by-term to find the coefficients. Substituting this result for A_1 in Equation (14) yields

$$\frac{l}{L} = 1 - \frac{1}{e}\left(1 - \frac{1 - e^{-(1+b)}}{a(1+b)(2+b)}\frac{1}{\xi}\right) + O(\xi^{-2}) \tag{17}$$

For $a < b$ this provides an expansion of the localisation length for low velocity, V, while for $a > b$ this provides an expansion for high velocity. These are the limits in which L tends to a constant, and Equation (17) gives the leading order correction, as well as the limiting value of l.

5.2. *Expansion for* $\xi \to 0$

In this limit, the cases $a < b$ and $a > b$ show different qualitative behaviour.

For $a < b$ we proceed by again writing A_1 as a series in ξ, making use of Equation (15.3.7) of [11], and then substituting this back into Equation (14) yields

$$\frac{l}{L} = \frac{1+a}{b-a}\left(e^{(b-a)/(1+a)} - 1\right)\left(\xi\frac{a(1+b)}{(1+a)}\right)^{1/(1+a)} + O\left(\left(\xi\frac{a(1+b)}{(1+a)}\right)^{\gamma}\right) \tag{18}$$

where $\gamma = \min\left((1+b)/(b-a),\, 2/(1+a)\right)$.

The above formula gives us an expansion of the localisation length for $V \to \infty$ only for the case $a < b$. Note that the leading power law is precisely what was derived from approximate arguments in Section 3, Equations (11) and (12).

For $a > b$ the solution of Equation (15) is, in general, singular. It may be found by setting $\tilde{v}(\tilde{x}) = 0$ for $\tilde{x} > c$, where c is to be determined. Recalling that $\mathrm{d}\tilde{v}/\mathrm{d}\tilde{x}$ is continuous, the solution can be found as

$$\tilde{v}(\tilde{x}) = \begin{cases} (1 - \frac{\tilde{x}}{c})^{(1+a)/(a-b)} & \text{for } 0 \le \tilde{x} < c, \\ 0 & \text{for } c \le \tilde{x} \le 1, \end{cases} \tag{19}$$

where c is now fixed in terms of ξ as $c = ((1+a)/(a-b))(\xi a((1+b)/(1+a)))^{1/1+a}$. It can be shown that the above solution is valid for $\xi < (1 + a/a(1+b))(a - b/1+a)^{1+a}$, which is equivalent to $c < 1$. The exact expression for the localisation length is then given by the leading term of Equation (18). This is again the same power law as in Equation (12). For $c > 1$, Equation (15) has a solution for A_1 in terms of ξ. A fuller analysis of the implications of the expansions (Equations (17) and (18)), will be given in a subsequent paper. Hence we see that the same power law is recovered in every case by detailed analysis, although the prefactor in Equation (12) remains an approximate estimate.

6. Conclusion

We have seen that an approximate derivation yields a power law dependence of localisation length upon boundary velocity, Equation (12), which turns out to be accurate and universal, in the sense that the same power law is found in exact analysis in the appropriate limits (for which localisation length is much smaller than the system size), for both of the distinct cases discussed here. This suggests that there may be some more rigorous general derivation to be found. In any case, this neat formula is obviously of great value for the analysis of further experimental results of the kind presented by Katgert et al. [8]. For the moment, such results seem likely to be confined to the case in which b is greater than a.

As yet, we cannot reconcile our results with the various experiments which exhibit quite different behaviours. This difficulty was already evident in the case which emerged from the work of Katgert et al. [8]. They commented that some of the earlier experiments involved less polydisperse specimens, which may account for different behaviour, specifically a V-independent localisation length. Alternatively, different surfactants or concentrations may have played a role, by giving rise to differing values of a and b. Further experiments are clearly called for, to resolve such questions.

Information concerning shear localisation may also be obtained from computer simulations of 2D foams based on a soft disk model, in which discrete elements (bubbles) interact with specific forces, as originally developed by Durian [12,13]. This will be the subject of a further paper [14].

Acknowledgements

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Note

1. www.wolfram.com/, www.maplesoft.com/.
   The power law which we have derived by various exact and approximate methods can also be deduced by a general dimensional argument.
   The localisation length l is a property of the solution of Equation (5), for boundary velocity V and boundary separation L. In the limit in which $l \to 0$, it cannot be dependent on L and hence must be a function of $(\frac{c_v}{c_d})$ and V only. Examination of the dimension of $(\frac{c_v}{c_d})$ as dictated by Equations (2) and (4) shows that in this case l must vary as in Equation (12), for dimensional consistency.

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